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Polarization effects in weak localization of light: Calculation of the copolarized and depolarized backscattering enhancement factors

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Weak localization of polarized light in a medium composed of arbitrary uncorrelated particles is considered. Saxon's reciprocity relation for the single-scattering amplitude matrix is used to derive a rigorous relation between the ladder-term and cyclical-term contributions to the Stokes reflection matrix in the pure backscattering direction. By solving numerically Chandrasekhar's vector radiative-transfer equation to compute the ladder-term contribution, the copolarized and depolarized backscattering enhancement factors are calculated for Rayleigh scattering and spherical latex particles in water. The computations for latex particles show good agreement with experimental data.

In recent years, considerable attention has been paid to the problem of enhanced backscattering of light from random media.[1] This enhanced backscattering is observed as a well-defined narrow peak in the angular distribution of the intensity of the scattered light at scattering angles near 180°. A well-known result of the theory of multiple scattering of *scalar* waves in discrete random media is that in the pure backscattering direction, the contribution of all the cyclical diagrams to the backscattered intensity is identical to that of all the ladder diagrams of orders $n \geq 2$.[2] Therefore, the backscattering coefficient γ is given by

$$\gamma=\gamma^1+\gamma^L+\gamma^C\equiv\gamma^1+2\gamma^L, \tag{1}$$

where γ^1 is the contribution of the first-order scattering, γ^L is the contribution of all the ladder diagrams of orders $n\geq 2$, and γ^C is the contribution of all the cyclical diagrams. An important consequence of Eq. (1) is the so-called "factor of two":

$$\zeta=\frac{\gamma^1+2\gamma^L}{\gamma^1+\gamma^L}\approx 2, \tag{2}$$

where ζ is the backscattering enhancement factor, defined as the ratio of the total backscattered intensity and the incoherent background intensity in the pure backscattering direction. The purpose of the present paper is to generalize Eq. (1) by taking into account the vector character of light. Also, assuming that the contribution of all the ladder diagrams can be found by solving the vector radiative-transfer equation, we calculate the copolarized and depolarized enhancement factors for several scattering models and compare these calculations with experimental data.

Consider a plane-parallel medium composed of arbitrary uncorrelated particles. To describe light scattering by a particle, we use a local right-handed Cartesian coordinate system, which has its origin inside the particle and a fixed orientation identical to that of the laboratory reference frame attached to the medium. The direction of light propagation is specified by a unit vector $\hat{\mathbf{n}}=(\theta,\phi)=\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}$, where θ is the polar angle, ϕ is the azimuth angle, and $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the corresponding unit vectors. Assume that the concentration of the particles is low and the particles may be considered independent scatterers. Thus, each particle can be specified by a (2×2) amplitude scattering matrix $\underline{F}(\hat{\mathbf{n}}',\hat{\mathbf{n}})$, which describes how θ and ϕ components of a plane wave $\mathbf{E}(\hat{\mathbf{n}})$, incident on the particle in the direction $\hat{\mathbf{n}}$, are transformed into θ and ϕ components of the wave $\mathbf{E}'(\hat{\mathbf{n}}')$, scattered by the particle in the far-field zone in the direction $\hat{\mathbf{n}}'$:[3,4]

$$\begin{pmatrix} E'_\theta \\ E'_\phi \end{pmatrix} \propto \underline{F}\begin{pmatrix} E_\theta \\ E_\phi \end{pmatrix}. \tag{3}$$

An important property of the amplitude scattering matrix is the reciprocity relation[5]

$$\underline{F}(-\hat{\mathbf{n}},-\hat{\mathbf{n}}')=\underline{Q}\,\underline{F}^T(\hat{\mathbf{n}}',\hat{\mathbf{n}})\underline{Q}\,, \tag{4}$$

where $\underline{Q}=\mathrm{diag}(1,-1)$, and T denotes matrix transpose. We emphasize here that the basis $\{\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}\}$ (instead of $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$) is used throughout the paper to describe the incident and scattered fields.

Let a plane wave $\mathbf{E}_0=\binom{E_{0\theta}}{E_{0\phi}}$ be incident upon the upper boundary of the medium in the direction $\hat{\mathbf{n}}_0$. Denote by $\underline{\rho}^0$ the corresponding density (or coherency) matrix which is defined by

$$\underline{\rho}^0=\mathbf{E}_0\mathbf{E}_0^{*T}=\begin{pmatrix} E_{0\theta}E_{0\theta}^* & E_{0\theta}E_{0\phi}^* \\ E_{0\phi}E_{0\theta}^* & E_{0\phi}E_{0\phi}^* \end{pmatrix}, \tag{5}$$

where the asterisk denotes complex conjugation. Let $\underline{\rho}$ be the density matrix of the light scattered by the medium in the far-field region in the direction $-\hat{\mathbf{n}}_0$. Transformation of the elements of the matrix $\underline{\rho}^0$ into those of the matrix $\underline{\rho}$

is given by

$$\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{21}\\ \rho_{22}\end{pmatrix} \propto \underline{\gamma}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)\begin{pmatrix}\rho_{11}^0\\ \rho_{12}^0\\ \rho_{21}^0\\ \rho_{22}^0\end{pmatrix}, \tag{6}$$

where $\underline{\gamma}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)$ is the (4×4) Stokes reflection matrix for the pure backscattering direction. Let us decompose the matrices $\underline{\rho}$ and $\underline{\gamma}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)$ as

$$\underline{\rho}=\underline{\rho}^1+\underline{\rho}^L+\underline{\rho}^C, \tag{7}$$

$$\underline{\gamma}=\underline{\gamma}^1+\underline{\gamma}^L+\underline{\gamma}^C, \tag{8}$$

where, as earlier, 1 denotes the contribution of the first-order scattering, L denotes the contribution of all the ladder diagrams of orders $n\geq 2$, and C denotes the contribution of all the cyclical diagrams. The problem is to express the elements of the matrix $\underline{\gamma}^C$ in those of the matrix $\underline{\gamma}^L$.

Denote by $(1,n)$ a "light path" formed by $n\geq 2$ arbitrary scattering centers, along which a wave travels, and by $(n,1)$ the time-reversed path, i.e., the path that is formed by the same scatterers, but along which the wave travels in the opposite direction. The waves that are scattered by the chains $(1,n)$ and $(n,1)$ in the pure backscattering direction $-\hat{\mathbf{n}}_0$ have equal phases and will constructively interfere. Denote by $\mathbf{E}^{(1,n)}$ and $\mathbf{E}^{(n,1)}$ the amplitudes of the two scattered waves and by $\underline{P}^{(1,n)}$ and $\underline{P}^{(n,1)}$ the corresponding (2×2) amplitude transformation matrices such that $\mathbf{E}^{(1,n)}\propto\underline{P}^{(1,n)}\mathbf{E}_0$ and $\mathbf{E}^{(n,1)}\propto\underline{P}^{(n,1)}\mathbf{E}_0$. The matrices $\underline{P}^{(1,n)}$ and $\underline{P}^{(n,1)}$ can be expressed in terms of products of the amplitude scattering matrices of the individual particles that enter the chains $(1,n)$ and $(n,1)$. Therefore, by using the single-scattering reciprocity relation, Eq. (4), one easily derives the reciprocity relation for the matrices $\underline{P}^{(1,n)}$ and $\underline{P}^{(n,1)}$:

$$\underline{P}^{(n,1)}=\underline{Q}(\underline{P}^{(1,n)})^T\underline{Q}. \tag{9}$$

Denote

$$P^{(1,n)}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}. \tag{10}$$

The contribution of the chains $(1,n)$ and $(n,1)$ to the matrix $\underline{\rho}^L$ is given by

$$\mathbf{E}^{(1,n)}(\mathbf{E}^{(1,n)})^{*T}+\mathbf{E}^{(n,1)}(\mathbf{E}^{(n,1)})^{*T},$$

while the contribution to the matrix $\underline{\rho}^C$ is given by

$$\mathbf{E}^{(1,n)}(\mathbf{E}^{(n,1)})^{*T}+\mathbf{E}^{(n,1)}(\mathbf{E}^{(1,n)})^{*T}.$$

The corresponding contributions to the matrices $\underline{\gamma}^L$ and $\underline{\gamma}^C$ are given, respectively, by

$$\begin{pmatrix}2aa^* & ab^*-ac^* & ba^*-ca^* & bb^*+cc^*\\ -ab^*+ac^* & 2ad^* & bc^*+cb^* & bd^*-cd^*\\ -ba^*+ca^* & bc^*+cb^* & 2da^* & db^*-dc^*\\ bb^*+cc^* & -bd^*+cd^* & -db^*+dc^* & 2dd^*\end{pmatrix} \tag{11}$$

and

$$\begin{pmatrix}2aa^* & ab^*-ac^* & ba^*-ca^* & -bc^*-cb^*\\ -ab^*+ac^* & 2ad^* & -bb^*-cc^* & bd^*-cd^*\\ -ba^*+ca^* & -bb^*-cc^* & 2da^* & db^*-dc^*\\ -bc^*-cb^* & -bd^*+cd^* & -db^*+dc^* & 2dd^*\end{pmatrix} \tag{12}$$

[see Eq. (10)]. By comparing the matrices (11) and (12), we have

$$\underline{\gamma}^C=\begin{pmatrix}\gamma_{11}^L & \gamma_{12}^L & \gamma_{13}^L & -\gamma_{32}^L\\ \gamma_{21}^L & \gamma_{22}^L & -\gamma_{41}^L & \gamma_{24}^L\\ \gamma_{31}^L & -\gamma_{41}^L & \gamma_{33}^L & \gamma_{34}^L\\ -\gamma_{32}^L & \gamma_{42}^L & \gamma_{43}^L & \gamma_{44}^L\end{pmatrix}. \tag{13}$$

This relation is the desired generalization to the vector case of the scalar identity $\gamma^C\equiv\gamma^L$.

For macroscopically isotropic media, the matrices $\underline{\gamma}^1$, $\underline{\gamma}^L$, and $\underline{\gamma}^C$ have the form

$$\begin{pmatrix}\gamma_{11} & 0 & 0 & \gamma_{14}\\ 0 & \gamma_{22} & \gamma_{23} & 0\\ 0 & \gamma_{32} & \gamma_{33} & 0\\ \gamma_{41} & 0 & 0 & \gamma_{44}\end{pmatrix}.$$

Therefore, for linearly polarized incident light with the density matrix components $\rho_{11}^0=1$, $\rho_{12}^0=\rho_{21}^0=\rho_{22}^0=0$, we may define the copolarized and depolarized backscattering enhancement factors as

$$\begin{aligned}\zeta_\parallel&=(\gamma_{11}^1+\gamma_{11}^L+\gamma_{11}^C)/(\gamma_{11}^1+\gamma_{11}^L)\\ &=(\gamma_{11}^1+2\gamma_{11}^L)/(\gamma_{11}^1+\gamma_{11}^L),\end{aligned} \tag{14}$$

$$\begin{aligned}\zeta_\perp&=(\gamma_{41}^1+\gamma_{41}^L+\gamma_{41}^C)/(\gamma_{41}^1+\gamma_{41}^L)\\ &=(\gamma_{41}^1+\gamma_{41}^L-\gamma_{32}^L)/(\gamma_{41}^1+\gamma_{41}^L).\end{aligned} \tag{15}$$

One sees that for the copolarized component, Eq. (14) is identical to Eq. (2) for scalar waves. At the same time, Eq. (15) for the depolarized component is quite different.

As is well known,[6] the Bethe-Salpeter equation under the ladder approximation of uncorrelated discrete scatterers results in the common radiative transfer equation. Therefore, assuming that the matrices $\underline{\gamma}^1$ and $\underline{\gamma}^L$ can be found by solving Chandrasekhar's[7] vector radiative-transfer equation, we used Eqs. (14) and (15) to compute the

TABLE I. Computed copolarized and depolarized backscattering enhancement factors and depolarization ratios for Rayleigh scattering and latex spherical particles in water.

Model	$\zeta_\parallel$	$\zeta_\perp$	χ
1	1.7521	1.1201	0.5167
2	1.92	1.10	0.69
3	2.00	1.26	0.96
4	1.82	1.12	0.57
5	1.97	1.17	0.88
6	1.98	1.21	0.92
7	2.00	1.25	0.95

copolarized and depolarized enhancement factors $\zeta_{\parallel}$ and $\zeta_{\perp}$ for several models. For simplicity, we assumed that the scattering medium is homogeneous and semi-infinite, and that the linearly polarized light is incident perpendicularly to the boundary of the medium. In addition to the enhancement factors, we computed the ratio

$$\chi=(\gamma_{41}^{1}+\gamma_{41}^{L})/(\gamma_{11}^{1}+\gamma_{11}^{L}), \tag{16}$$

which describes the depolarization of the incoherent background.

The incoherent Stokes reflection matrix of a semi-infinite homogeneous medium may be efficiently computed by using the doubling method[8] or by solving numerically Ambartsumian's[9] nonlinear integral equation for the reflection matrix

$$\begin{aligned}(-u+u_0)\underline{R}(u,\phi;u_0,\phi_0)={}&\frac{w}{4}\underline{Z}(u,\phi;u_0,\phi_0)-\frac{wu}{4\pi}\int_0^1 du'\int_0^{2\pi}d\phi'\,\underline{R}(u,\phi;u',\phi')\underline{Z}(u',\phi';u_0,\phi_0)\\&+\frac{wu_0}{4\pi}\int_0^1 du'\int_0^{2\pi}d\phi'\,\underline{Z}(u,\phi;-u',\phi')\underline{R}(-u',\phi';u_0,\phi_0)-\frac{wuu_0}{4\pi^2}\int_0^1 du'\int_0^{2\pi}d\phi'\int_0^1 du''\\&\times\int_0^{2\pi}d\phi''\,\underline{R}(u,\phi;u',\phi')\underline{Z}(u',\phi';-u'',\phi'')\underline{R}(-u'',\phi'';u_0,\phi_0),\end{aligned} \tag{17}$$

where the Z axis of the coordinate system is assumed to coincide with the inward normal to the boundary of the semi-infinite medium, $u=\cos\theta$, $u_0=\cos\theta_0$, w is the single-scattering albedo, $\underline{Z}(\hat{\mathbf{n}},\hat{\mathbf{n}}')$ is the phase matrix, and $\underline{R}(\hat{\mathbf{n}},\hat{\mathbf{n}}_0)$ with $-1\le u\le 0$ and $0\le u_0\le 1$ is the incoherent reflection matrix such that

$$\underline{\gamma}^{1}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)+\underline{\gamma}^{L}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)=\underline{R}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)$$

and

$$\underline{\gamma}^{1}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)=w\underline{Z}(-\hat{\mathbf{n}}_0,\hat{\mathbf{n}}_0)/(8u_0).$$

In our computations, we used the latter approach. Specifically, the method of iterations was employed to solve Eq. (17). For nearly conservative scattering ($1-w\ll 1$), simple iterations converge very slow. To accelerate convergence, de Rooij's[10] numerical procedure was exploited.

Seven models of the scattering medium were considered which are specified as follows. Model 1 is Rayleigh scattering with the single-scattering albedo $w=1$. Models 2 to 7 are latex spherical particles in water.[11,12] The relative refractive index is 1.194, and the wavelength is 0.476 μm for models 2 and 3 and 0.387 μm for models 4 to 7. Diameters of the particles are, respectively, 0.215, 1.091, 0.109, 0.305, 0.46, and 0.797 μm. The results of the computations are given in Table I.

The depolarized backscattering enhancement factor for Rayleigh scattering was calculated earlier by Stephen and Cwilich,[13] van Albada and Lagendijk,[14] and Akkermans *et al.*[15] Stephen and Cwilich used diffusion approximation and obtained $\zeta_{\perp}=1.17$. van Albada and Lagendijk and Akkermans *et al.* used numerical simulation of multiple scattering of polarized light and obtained $\zeta_{\perp}=1.11$ and 1.14, respectively. All these estimates are in good agreement with our result $\zeta_{\perp}=1.1201$. Agreement between the values $\zeta_{\parallel}=1.88$ and $\chi=0.42$, obtained by Stephen and Cwilich, and our values $\zeta_{\parallel}=1.7521$ and $\chi=0.5167$ is also satisfactory.

Our computations for models 2 to 7 are in reasonable agreement with experimental results of van Albada, van der Mark, and Lagendijk[11] and Wolf *et al.*[12] In particular, depolarized enhancement factors for all the particles, partial retention of incoherent background polarization for small particles (models 2 and 4), and almost complete depolarization of incoherent background by larger particles (models 3 and 5 to 7) are satisfactorily reproduced. Experimental values of the copolarized backscattering enhancement factor seem to be systematically lower than the theoretically computed ones. For a discussion of possible explanations of this phenomenon, we refer the reader to van Albada, van der Mark, and Lagendijk.[16]

In summary, the model of uncorrelated discrete scatterers was used to derive rigorous relation between the ladder-term and cyclical-term contributions to the Stokes reflection matrix in the pure backscattering direction. The numerical solution of Chandrasekhar's vector radiative-transfer equation was used to calculate the incoherent part of the Stokes backscattering matrix. Copolarized and depolarized backscattering enhancement factors as well as a parameter describing depolarization of the incoherent background were computed for Rayleigh scattering and latex spherical particles in water. Numerical results for latex particles show reasonable agreement with experimental data of van Albada, van der Mark, and Lagendijk[11] and Wolf *et al.*[12]

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